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Refined formulation of quantum analysis, q-derivative and exponential splitting

Masuo Suzuki

Department of Applied Physics, Tokyo University of Science, 1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

E-mail: msuzuki@rs.kagu.tus.ac.jp

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Abstract

Quantum analysis is reformulated to clarify its essence, namely the invariance of quantum derivative for any choice of definitions of the differential $\mathrm{d}f(A)$ satisfying the Leibniz rule. This formulation with use of the inner derivation δ_A is convenient to study quantum corrections in contrast to the Feynman operator calculus. The present analysis can also be used to find a general scheme of constructing exponential product formulae of higher order. General recursive schemes are also reviewed with an emphasis to standard symmetric splitting formulae. Multiple integral representations of q-derivatives are derived using such general integral formulae of quantum derivatives as are expressed by hyperoperators. A simple explanation of the connection between quantum derivatives and q-derivative is also given.

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1. Introduction

We have often to evaluate, in modern physics, commutators such as $[f(A), B]$ or more generally $[f(A), g(B)]$ for arbitrary functions $f(x)$ and $g(x)$ and to expand a function $f(A + B)$ with respect to B for non-commutable operators A and B. Such problems have already been studied by many authors [1–9] using Feynman's indices [10].

The same problems have also been studied by the present author [11–16] in a different viewpoint using the inner derivation δ_A defined by $\delta_A B = [A, B] = AB - BA$, in order to clarify the crossover from the classical to quantum derivatives. This quantum analysis is also useful [17–29] in constructing exponential product formulae of higher order. A similarity between quantum analysis and q-derivative is also discussed.

2. Refined formulation of quantum analysis

The essence of the ordinary differential calculus is manifested in the following Taylor expansion formula,

$$f(x+h) = f(x) + f^{(1)}(x)h + \cdots + \frac{f^{(n)}(x)}{n!}h^n + \cdots, \tag{2.1}$$

where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$. Even the definition of differentiation is included in equation (2.1), as is easily seen.

Now we discuss operator functions such as $f(A)$ and $f(A+xB)$. If the operators A and B commute with each other, namely $[A, B] = AB - BA = 0$, then we have

$$f(A+xB) = f(A) + xf^{(1)}(A)B + \cdots + \frac{x^n}{n!}f^{(n)}(A)B^n + \cdots, \tag{2.2}$$

where $f^{(n)}(A)$ is an operator function obtained by replacing the c-number variable x with the operator A in the ordinary nth derivative $f^{(n)}(x)$.

The purpose of the present paper is to give a refined formulation of quantum analysis proposed by the present author [11–16] in a general situation where A and B do not commute with each other. We try to expand the operator function $f(A+xB)$ in a power series of x as follows:

$$f(A+xB) = \sum_{n=0}^{\infty} a_n(A, B)x^n. \tag{2.3}$$

Clearly, the coefficient operator $a_n(A, B)$ is given by

$$a_n(A, B) = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}x^n}f(A+xB)\right]_{x=0}. \tag{2.4}$$

Each $a_n(A, B)$ is composed of many products of A and B of different order, and it seems to be very complicated to treat analytically because of the noncommutativity of A and B. Our quantum analysis gives a convenient expression [11–16] to each $a_n(A, B)$ in terms of the commutable hyperoperators δ_A and L_A $(=A)$ defined by $L_A Q = AQ$. This is performed by expressing $a_n(A, B)$ in the form

$$a_n(A, B) = \hat{f}_n(A, \{\delta_j\})B^n, \tag{2.5}$$

where the hyperoperator δ_j is defined by

$$\delta_j B^n = B^{j-1}(\delta_A B)B^{n-j} \tag{2.6}$$

for $j = 1, 2, \ldots, n$. The operator A in $\hat{f}_n(A, \{\delta_j\})$ is interpreted as such a hyperoperator as multiplies A to B^n from the left-hand side (namely AB^n). Note that A and δ_A commute with each other. This is the reason why the hyperoperator $\hat{f}_n(A, \{\delta_j\})$ is convenient to treat analytically once it has been obtained explicitly [11–16]. Furthermore, we introduce the concept of quantum derivative, namely the quantum derivative of $f(A)$ with respect to the operator A itself, and we write it as $\mathrm{d}f(A)/\mathrm{d}A$ analogously to the ordinary derivative $\mathrm{d}f(x)/\mathrm{d}x$. However, the meaning of the former is quite different from the latter, namely it is a mapping operator (or hyperoperator) to map B^n to the operator $a_n(A, B)$ defined by equation (2.4). Thus, our quantum derivative or hyperoperator $\mathrm{d}^n f(A)/\mathrm{d}A^n$ is defined as

$$\frac{\mathrm{d}^n f(A)}{\mathrm{d}A^n} \equiv n!\hat{f}_n(A, \{\delta_j\}). \tag{2.7}$$

In order to find an explicit and compact expression of $\hat{f}_n(A, \{\delta_j\})$ or $\mathrm{d}^n f(A)/\mathrm{d}A^n$, we start with a simple operator function $f(A) = A^{m+n}$ where m is a positive integer. First note that for $f(A) = A^{m+n}$ we have

$$\begin{aligned} a_n(A, B) &= \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}x^n}(A + xB)^{m+n}\right]_{x=0} \\ &= \sum_{k_1+\cdots+k_{n+1}=m, k_j \geqslant 0} A^{k_1} B A^{k_2} \cdots A^{k_n} B A^{k_{n+1}} \\ &\equiv \{A^m B^n\}_{\mathrm{sym}(A,B)}. \end{aligned} \tag{2.8}$$

Here, $\{A^m B^n\}_{\mathrm{sym}(A,B)}$ denotes the symmetrized product. This sum of products is easily expressed using Feynman's indices in the Feynman operator calculus [9, 10]. However, it is inconvenient to study quantum effects. One of the purposes of our quantum analysis is to make such a formulation able to express quantum corrections explicitly. One of the key formulae for this purpose in our quantum analysis is the following [15]:

$$\{A^m B^n\}_{\mathrm{sym}(A,B)} = \frac{(m+n)!}{m!}\int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \left(A - \sum_{j=1}^n t_j \delta_j\right)^m B^n. \tag{2.9}$$

This can be proved by mathematical induction. For $n = 1$, we have

$$\{A^m B\}_{\mathrm{sym}(A,B)} = (m+1)\int_0^1 \mathrm{d}t\,(A - t\delta_A)^m B, \tag{2.10}$$

because the right-hand side of equation (2.10) is integrated as

$$\begin{aligned} [-(A - t\delta_A)^{m+1}/\delta_A]_0^1 B &= [(A^{m+1} - (A - \delta_A)^{m+1})/\delta_A]B \\ &= \sum_{k=0}^m A^{m-k}(A - \delta_A)^k B = \{A^m, B\}_{\mathrm{sym}(A,B)}. \end{aligned} \tag{2.11}$$

Here, the symbol $1/\delta_A$ seems to be strange at a glance, because δ_A^{-1} does not exist. However, the numerator $(A^{m+1} - (A - \delta_A))^{m+1}$ contains the factor δ_A and consequently the ratio of these two hyperoperators can be defined uniquely. Here we have also used the following simple relation [11]:

$$(A - \delta_A)^k B = BA^k \tag{2.12}$$

for any positive integer k, or more generally

$$f(A - \delta_A)B = Bf(A) \qquad \text{or} \qquad \delta_{f(A)} = f(A) - f(A - \delta_A) \tag{2.13}$$

for any analytic function $f(x)$.

If we assume that equation (2.9) holds for $n - 1$, then it is shown to hold also for n, using the following important transformation formula [16]:

$$\begin{aligned} &Bf_1(A)Bf_2(A)\cdots Bf_n(A) \\ &\quad = f_1(A - \delta_1)f_2(A - \delta_1 - \delta_2)\cdots f_n(A - \delta_1 - \delta_2 - \cdots - \delta_n)B^n \end{aligned} \tag{2.14}$$

for any analytic functions $\{f_j(x)\}$.

Thus, we find that for $f(A) = A^{m+n}$ we have

$$\begin{aligned} a_n(A, B) &= \hat{f}_n(A, \{\delta_j\})B^n \\ &= \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n f^{(n)}\left(A - \sum_{j=1}^n t_j\delta_j\right)B^n. \end{aligned} \tag{2.15}$$

Therefore, the above relation (2.15) holds for any analytic function $f(x)$, namely we arrive finally at

$$\frac{\mathrm{d}^n f(A)}{\mathrm{d}A^n} = n! \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n\, f^{(n)}\left(A - \sum_{j=1}^{n} t_j \delta_j\right) \tag{2.16}$$

or

$$f(A + xB) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{\mathrm{d}^n f(A)}{\mathrm{d}A^n} B^n. \tag{2.17}$$

This is one of our main formulae in quantum analysis. If A commutes with B, then $\{\delta_j\}$ in the above formula are unnecessary. Thus, equation (2.17) together with equation (2.16) is reduced to equation (2.2) in this case. The convergence of the above expansion is proved in the Banach space [11] and for some restricted unbounded operators [14].

The difference between the present formulation and the Feynman operator calculus [9] is slight in the first-order derivative but it is substantial in higher-order derivatives, as is seen from the above general formula (2.16).

3. Generalization of quantum analysis

In the preceding section, we have studied the operator Taylor expansion of $f(A + xB)$ and have defined the nth quantum derivative $\mathrm{d}^n f(A)/\mathrm{d}^n A$. However, there are many other ways to define them. As in [1], we may start with the following Gâteau differential:

$$\mathrm{d}f(A) = \lim_{h\to 0} \frac{f(A + h\,\mathrm{d}A) - f(A)}{h}, \tag{3.1}$$

for any operator $\mathrm{d}A$ or with the commutator

$$\mathrm{d}f(A) = [H, f(A)] \tag{3.2}$$

for a certain fixed operator H, as in [3]. It is well known that the above two differentials satisfy the Leibniz rule

$$\mathrm{d}(f(A)g(A)) = (\mathrm{d}f(A))g(A) + f(A)\,\mathrm{d}g(A). \tag{3.3}$$

Then, it is generally shown [16] that we have

$$\mathrm{d}f(A) = \frac{\delta_{f(A)}}{\delta_A}\,\mathrm{d}A \qquad \text{namely} \quad \frac{\mathrm{d}f(A)}{\mathrm{d}A} = \frac{\delta_{f(A)}}{\delta_A} \tag{3.4}$$

because

$$\mathrm{d}(Af(A)) = \mathrm{d}(f(A)A), \tag{3.5}$$

namely

$$(\mathrm{d}A)f(A) + A\,\mathrm{d}f(A) = (\mathrm{d}f(A))A + f(A)\,\mathrm{d}A \tag{3.6}$$

owing to the Leibniz rule (3.3). This yields

$$[A, \mathrm{d}f(A)] = [f(A), \mathrm{d}A] \tag{3.7}$$

or

$$\delta_A\,\mathrm{d}f(A) = \delta_{f(A)}\,\mathrm{d}A. \tag{3.8}$$

It is instructive to remark that the quantum derivative $\mathrm{d}f(A)/\mathrm{d}A$ is expressed by the ratio

$$\frac{\delta_{f(A)}}{\delta_A} = \frac{(f(A) - f(A - \delta_A))}{\delta_A} = \int_0^1 f^{(1)}(A - t\delta_A)\,\mathrm{d}t \tag{3.9}$$

because the above second expression is denoted by the difference of hyperoperators and because its connection with the ordinary (or classical) derivative is transparent. (In fact, it is reduced to $f^{(1)}(A)$ in the limit $\delta_A \to 0$.)

Similarly for higher-order quantum derivatives, we can repeat the above procedure. Namely from the identity $\mathrm{d}^n(f(A)g(A)) = \mathrm{d}^n(g(A)f(A))$, we obtain the following formula:

$$\mathrm{d}^n f(A) = \frac{\mathrm{d}^n f(A)}{\mathrm{d}A^n}(\mathrm{d}A)^n \tag{3.10}$$

with the same expression (2.16) for any definition of the quantum differential $\mathrm{d}f(A)$. This shows that the quantum derivative is invariant [16] for any choice of definitions of the differential satisfying the Leibniz rule, though $\mathrm{d}^n f(A)$ and $(\mathrm{d}A)^n$ depend on the definitions.

4. Applications of quantum analysis

A simple application of quantum analysis is to evaluate the commutator $[f(A), g(B)]$ for arbitrary analytic functions $f(x)$ and $g(x)$. It is expressed in the form

$$[f(A), g(B)] = \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}t f^{(1)}(A - s\delta_A) g^{(1)}(B - t\delta_B)[A, B]. \tag{4.1}$$

This is derived as follows:

$$\begin{aligned}[f(A), g(B)] &= \delta_{f(A)} g(B) = \frac{\mathrm{d}f(A)}{\mathrm{d}A}\delta_A g(B) \\ &= \frac{\mathrm{d}f(A)}{\mathrm{d}A}\delta_{g(B)}(-A) = \frac{\mathrm{d}f(A)}{\mathrm{d}A}\frac{\mathrm{d}g(B)}{\mathrm{d}B}\delta_B(-A) \\ &= \frac{\mathrm{d}f(A)}{\mathrm{d}A}\frac{\mathrm{d}g(B)}{\mathrm{d}B}[A, B],\end{aligned} \tag{4.2}$$

where we have used formulae (3.4) and (3.5).

In particular, we have

$$[\mathrm{e}^{xA}, \mathrm{e}^{yB}] = \int_0^x \mathrm{d}s \int_0^y \mathrm{d}t\, \mathrm{e}^{(x-s)A}\, \mathrm{e}^{(y-t)B}[A, B]\, \mathrm{e}^{tB}\, \mathrm{e}^{sA}, \tag{4.3}$$

where we have made use of the identity

$$\mathrm{e}^{x\delta_A} Q = \mathrm{e}^{xA} Q\, \mathrm{e}^{-xA}. \tag{4.4}$$

Equation (4.3) yields Kubo's identity [33]

$$[\mathrm{e}^{-\beta\mathcal{H}}, A] = \int_0^\beta \mathrm{d}\lambda\, \mathrm{e}^{-(\beta-\lambda)\mathcal{H}}[A, \mathcal{H}]\, \mathrm{e}^{-\lambda\mathcal{H}} \tag{4.5}$$

by putting $x = 0$ after the differentiation of both sides of equation (4.3) with respect to x. This identity is rewritten as

$$[\mathrm{e}^{-\beta\mathcal{H}}, A] = \mathrm{i}\hbar \int_0^\beta \mathrm{e}^{-\beta\mathcal{H}} \dot{A}(-\mathrm{i}\hbar\lambda)\, \mathrm{d}\lambda \tag{4.6}$$

with

$$\dot{A}(t) = \mathrm{e}^{it\mathcal{H}/\hbar} \dot{A}\, \mathrm{e}^{-it\mathcal{H}/\hbar} \qquad \text{and} \qquad \dot{A} = \frac{\mathrm{i}}{\hbar}[\mathcal{H}, A] \tag{4.7}$$

for the Hamiltonian $\mathcal{H}$ of the relevant system. An alternative interpretation of equation (4.5) yields

$$\mathrm{d}\,\mathrm{e}^{-\beta\mathcal{H}} = -\int_0^\beta \mathrm{d}\lambda\, \mathrm{e}^{-(\beta-\lambda)\mathcal{H}}(\mathrm{d}\mathcal{H})\, \mathrm{e}^{-\lambda\mathcal{H}} \tag{4.8}$$

through definition (3.2). The above relation (4.6) plays an important role in the linear response theory [33].

At a glance, the right-hand side of equation (4.3) seems to be more complicated than the left-hand side. However, this transformation is sometimes useful, for example, in evaluating the norm of $[\mathrm{e}^{xA}, \mathrm{e}^{yB}]$, because the right-hand side of equation (4.3) is a single product of the commutator $[A, B]$ with the two types of exponential operators, whose norm is easily evaluated.

Another example is to extend [13, 27] the Baker–Campbell–Hausdorff (BCH) formula. The original BCH formula is given in the form

$$\mathrm{e}^{xA}\,\mathrm{e}^{xB} = \exp\left(x(A+B) + \tfrac{1}{2}x^2[A,B] + \cdots\right) \tag{4.9}$$

The essence of the BCH formula is that the exponential part of equation (4.9) is a linear combination of $(A+B)$ and commutators of A and B. Our quantum analysis is useful in studying the operator function $\Phi(x)$ defined in

$$\mathrm{e}^{A_1(x)}\,\mathrm{e}^{A_2(x)}\cdots\mathrm{e}^{A_r(x)} = \mathrm{e}^{\Phi(x)} \tag{4.10}$$

for an arbitrary set of operators $\{A_j(x)\}$ and for an arbitrary positive integer r. Here we assume that $A_j(0) = 0$ for all j and consequently that $\Phi(0) = 0$. As a generalization of the BCH formula, the operator function $\Phi(x)$ in equation (4.10) is expressed [13] in terms of $\{A_j(x)\}$ and their commutators (free Lie elements). By differentiating both sides of equation (4.10), we obtain

$$\frac{\mathrm{d}\,\mathrm{e}^{\Phi(x)}}{\mathrm{d}\Phi(x)}\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x} = \sum_{j=1}^{r}\mathrm{e}^{A_1(x)}\cdots\left(\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{e}^{A_j(x)}\right)\cdots\mathrm{e}^{A_r(x)}. \tag{4.11}$$

Here, the quantum derivative $\mathrm{d}\,\mathrm{e}^{\Phi(x)}/\mathrm{d}\Phi(x)$ denotes a hyperoperator given by [11–16]

$$\frac{\mathrm{d}\,\mathrm{e}^{\Phi(x)}}{\mathrm{d}\Phi(x)} = \mathrm{e}^{\Phi(x)}\Delta(-\Phi(x)) \qquad \text{and} \qquad \Delta(A) = \frac{\mathrm{e}^{\delta_A}-1}{\delta_A}. \tag{4.12}$$

After some calculations, we finally arrive at

$$\begin{aligned}\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x} &= \Delta^{-1}(-\Phi(x))\sum_{j=1}^{r}\exp(-\delta_{A_r(x)})\cdots\exp(-\delta_{A_{j+1}(x)})\Delta(-A_j(x))\frac{\mathrm{d}A_j(x)}{yx}\\ &= \Delta^{-1}(-\Phi(x))\sum_{j=1}^{r}\exp(\delta_{A_1(x)})\cdots\exp(\delta_{A_{j-1}(x)})\Delta(A_j(x))\frac{\mathrm{d}A_j(x)}{\mathrm{d}x},\end{aligned} \tag{4.13}$$

with $\Delta(-A) = \mathrm{e}^{-\delta_A}\Delta(A)$ and

$$\Delta^{-1}(\Phi(x)) = \frac{\delta_{\Phi(x)}}{\exp\delta_{\Phi(x)} - 1}. \tag{4.14}$$

Here we have used the specific relation

$$\Delta^{-1}(-\Phi) = \Delta^{-1}(\Phi)\exp(\delta_{A_1})\cdots\exp(\delta_{A_r}), \tag{4.15}$$

which is valid only for the operator function Φ defined in equation (4.10).

Since $\Delta^{-1}(\Phi(x))$ is expressed in terms of $\delta_{\Phi(x)}$ through relation (4.14) and in turn $\delta_{\Phi(x)}$ is expressed by

$$\delta_{\Phi(x)} = \log\mathrm{e}^{\delta_{\Phi(x)}} = \log[\exp(\delta_{A_1(x)})\cdots\exp(\delta_{A_r(x)})], \tag{4.16}$$

we obtain the solution of the differential equation with the initial condition $\Phi(0) = 0$ in the form [3]

$$\Phi(x) = \sum_{j=1}^{r}\int_0^x h(t)\exp(\delta_{A_1(t)})\cdots\exp(\delta_{A_{j-1}(t)})\Delta(A_j(t))\frac{\mathrm{d}A_j(t)}{\mathrm{d}t}\,\mathrm{d}t, \tag{4.17}$$

where

$$h(t) = \frac{\log[\exp(\delta_{A_1(t)})\exp(\delta_{A_2(t)})\cdots\exp(\delta_{A_r(t)})]}{\exp(\delta_{A_1(t)})\exp(\delta_{A_2(t)})\cdots\exp(\delta_{A_r(t)})-1}$$
$$= \sum_{n=1}^{\infty}\frac{1}{n}[1-\exp(\delta_{A_1(t)})\exp(\delta_{A_2(t)})\cdots\exp(\delta_{A_r(t)})]^{n-1} \qquad (4.18)$$

This is a very useful formula in practical applications. In fact, when $A_1(x) = xA_1 = xA$ and $A_2(x) = xA_2 = xB$ (for $r = 2$), we obtain [13]

$$\Phi(x) \equiv \log(\mathrm{e}^{xA}\,\mathrm{e}^{xB}) = \sum_{n=1}^{\infty}\frac{1}{n}\int_0^x (1-\mathrm{e}^{t\delta_A}\mathrm{e}^{t\delta_B})^{n-1}(A+\mathrm{e}^{t\delta_A}B)\,\mathrm{d}t. \qquad (4.19)$$

This yields an explicit expression of the BCH formula, which is useful in studying exponential product formulae of higher order, as will be discussed in the succeeding section. The above formulae (4.17) and (4.19) show explicitly that the exponential part $\Phi(x)$ is a linear combination of $\{A_j\}$ and their commutators, as was proved by Baker and Hausdorff. In other words, the above formulation gives an alternative proof of their theorem.

5. Exponential splitting formulae of higher order

As was discussed systematically by the present author and his collaborators [20–24, 26, 27], the exponential operator $\mathrm{e}^{x(A+B)}$ is split in the form

$$\mathrm{e}^{x(A+B)} = f_m(xA, xB) + \mathrm{O}(x^{m+1}), \qquad (5.1)$$

with

$$f_m(xA, xB) = \mathrm{e}^{t_1xA}\,\mathrm{e}^{t_2xB}\,\mathrm{e}^{t_3xA}\,\mathrm{e}^{t_4xB}\cdots\mathrm{e}^{t_MxA}. \qquad (5.2)$$

The splitting parameters $\{t_j\}$ can be determined by the requirement that

$$\Phi_m(x, \{t_j\}, A, B) \equiv \log f_m(xA, xB) = x(A+B) + \mathrm{O}(x^{m+1}) \qquad (5.3)$$

using formula (4.16) or an extension of (4.18) to general r, namely

$$\Phi(x) \equiv \log(\mathrm{e}^{xA_1}\,\mathrm{e}^{xA_2}\cdots\mathrm{e}^{xA_r})$$
$$= \sum_{n=1}^{\infty}\sum_{j=1}^{r}\frac{1}{n}\int_0^x \mathrm{d}t\,(1-\exp(t\delta_{A_1})\cdots\exp(t\delta_{A_r})]^{n-1}$$
$$\times\exp(t\delta_{A_1})\cdots\exp(t\delta_{j-1})A_j. \qquad (5.4)$$

For explicit calculations of $\{t_j\}$, see [11–16].

An alternative method to find higher-order splitting formulae is to make use of the recursive scheme discovered by the present author [17].

The mth-order formula $f_m(xA, xB)$ is constructed recursively by the product of the form

$$f_m(xA, xB) = f_{m-1}(p_1xA, p_1xB)f_{m-1}(p_2xA, p_2xB)\cdots f_{m-1}(p_rxA, p_rxB). \qquad (5.5)$$

under the conditions [17]

$$p_1 + p_2 + \cdots + p_r = 1, \qquad (5.6a)$$

and

$$p_1^m + p_2^m + \cdots + p_r^m = 0. \qquad (5.6b)$$

The above second equation (5.6b) is easily derived from the observation that the lowest correction term of the product (5.5) to the true exponential operator $\exp(x(A+B))$ is proportional to the sum $\left(p_1^m + p_2^m + \cdots + p_r^m\right)$.

For example, we have the following so-called standard higher-order recursive symmetric formula [17, 18]

$$S_{2m}(x) = S_{2m-2}^2(p_{2m}x)S_{2m-2}((1-4p_{2m})x)S_{2m-2}^2(p_{2m}x) \tag{5.7}$$

with $S_2(x) = \mathrm{e}^{\frac{x}{2}A}\,\mathrm{e}^{xB}\,\mathrm{e}^{\frac{1}{2}xA}$ and

$$p_{2m} = \frac{1}{4-4^{1/(2m-1)}}. \tag{5.8}$$

This is called a zig-zag decomposition [17–20].

6. Multiple integral representations of q-derivatives

Euler's identities, the Jacobi identity, Heine's formula and the Ramanujan formula have been effectively used in mathematical physics [9]. The following q-derivative is useful in studying these identities and formulae.

As was shown in equation (3.4), the quantum derivative $\mathrm{d}f(A)/\mathrm{d}A$ is expressed by the ratio of the hyperoperators $\delta_{f(A)}$ and δ_A, where $\delta_{f(A)} = f(A) - f(A-\delta_A)$. It should be remarked that $\delta_{f(A)}$ is expressed by the difference of two hyperoperators. This yields the noncommutativity effect of A and $\mathrm{d}A$. This structure of difference reminds us of the q-derivative D_q defined by [34]

$$D_q f(x) = \frac{f(qx)-f(x)}{(q-1)x} \equiv \frac{\mathrm{d}_q f(x)}{\mathrm{d}_q x} \tag{6.1}$$

for an ordinary function $f(x)$. Clearly we have

$$D_{q\to 1} f(x) = f^{(1)}(x), \tag{6.2}$$

when $f(x)$ is analytic. It is easy to show that

$$D_q^2 f(x) = \frac{f(q^2x) - (q+1)f(qx) + qf(x)}{q(q-1)^2x^2}. \tag{6.3}$$

An explicit expression of $D_q^n f(x)$ for an arbitrary positive integer n seems to be rather complicated. However, we have clearly

$$D_{q\to 1}^n f(x) = f^{(n)}(x) \tag{6.4}$$

by definition (6.1), as it should be. Thus, there is a certain possibility of expressing $D_q^n f(x)$ in terms of $f^{(n)}(x)$. In order to solve this problem, our general multiple integral representation of $\mathrm{d}^n f(A)/\mathrm{d}A^n$ in equation (2.16) is very suggestive. In fact, after some considerations, we obtain the following desired expression:

$$D_q^n f(x) = [n]_q! \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n\, f^{(n)}(\{1 + (q-1)(t_1 + t_2 q + \cdots + t_n q^{n-1})\}x), \tag{6.5}$$

where

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1} \tag{6.6}$$

and

$$[n]_q! = [1]_q \times [2]_q \times \cdots \times [n]_q. \tag{6.7}$$

For example, we have

$$D_q f(x) = \int_0^1 f^{(1)}((1+(q-1)t)x)\,\mathrm{d}t, \tag{6.8}$$

which is easily integrated to give the result (6.1). Furthermore, we have

$$D_q^2 f(x) = [2]_q \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}t'\, f^{(2)}((1+(q-1)(t+qt')x), \tag{6.9}$$

which gives again the result (6.3). In principle, we can perform this procedure. However, it is convenient for proving the general expression (6.5) to make use of mathematical induction as follows. We assume that equation (6.5) holds for $n-1$. Then, we obtain

$$\begin{aligned} D_q^n f(x) = [n-1]_q! &\int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-2}} \mathrm{d}t_{n-1} \\ &\times D_q f^{(n-1)}(\{1+(q-1)(t_1+t_2q+\cdots+t_{n-1}q^{n-2})\}x) \\ = [n-1]_q! &\int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-2}} \mathrm{d}t_{n-1} \times \frac{1}{(q-1)x} \\ &\times [f^{(n-1)}(\{1+(q-1)(t_1+t_2q+\cdots+t_{n-1}q^{n-2})\}qx) \\ &- f^{(n-1)}(\{1+(q-1)(t_1+t_2q+\cdots+t_{n-1}q^{n-2})\}x)] \\ = [n]_q! &\int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \\ &\times f^{(n)}(\{1+(q-1)(t_1+t_2q+\cdots+t_nq^{n-1})\}x). \end{aligned} \tag{6.10}$$

The last equality in equation (6.10) is derived by putting $t = 1, x_1 = (q-1)x, x_2 = (q-1)qx, \ldots, x_n = (q-1)q^{n-1}x$ in the following relation [6]:

$$\begin{aligned} (x_1+\cdots+x_n) &\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n\, f^{(m+1)}\left(tx+\sum_{j=1}^n t_jx_j\right) \\ &= \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-2}} \mathrm{d}t_{n-1} \\ &\times \left(f^{(m)}\left(t(x+x_1)+\sum_{j=1}^{n-1} t_jx_{j+1}\right) - f^{(m)}\left(tx+\sum_{j=1}^{n-1} t_jx_j\right)\right), \end{aligned} \tag{6.11}$$

which holds for any positive integers m and n, for arbitrary variables $\{x_j\}$, and for an arbitrary analytic function $f(x)$. A simple proof of equation (6.11) is given by differentiating $f_n(t)$, which is defined by the left-hand side of equation (6.11) minus the right-hand side and by noting that $f_n(0) = 0$ and $f_n'(t) \equiv 0$, which is derived by assuming $f_{n-1}(t) \equiv 0$. This gives $f_n(t) \equiv 0$ by mathematical induction, using $f_1(t) \equiv 0$. It is evident that equation (6.5) holds in the case $n = 1$. Thus, we finally arrive at the general formula (6.5) by mathematical induction. The relation between q-derivatives and ordinary higher-order derivatives $f^{(n)}(x)$ is transparent in representation (6.5). That is, we have equation (6.4) by putting $q = 1$ in equation (6.5).

It will also be instructive to discuss here explicitly the relation between the quantum Taylor expansion formula (2.17) with equation (2.16) and the general higher-order q-derivative (6.5).

For this purpose, we study the operator function $f(A+xB)$ when $BA = qAB$ for a number q commuting with both A and B. Noting that $\delta_A B = (1-q)AB$ in this case and consequently that

$$\delta_j B^n = B^{j-1}(\delta_A B)B^{n-j} = (1-q)q^{j-1}AB^n, \tag{6.12}$$

we obtain, from (2.16),

$$\frac{1}{n!}\frac{\mathrm{d}^n f(A)}{\mathrm{d}A^n}B^n = \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n f^{(n)}(\{1+(q-1)(t_1+t_2 q+\cdots+t_n q^{n-1})\}A)B^n$$
$$= \frac{1}{[n]_q!}(D_q^n f(A))B^n \tag{6.13}$$

using the general formula (6.5) for $x \to A$ (operator). In particular, if $q = 1$ (namely, A and B are commutable), then we obtain the expansion formula (2.2), using $[n]_{q=1} = n$ and equation (6.2), as it should be.

Now we give here a simple explanation both for the validity of the general multiple integral representation (6.5) of $D_q^n f(x)$ and for the specific relation (6.13) in the case $BA = qAB$.

First we rewrite $D_q f(A)$ and $\mathrm{d}f(A)/\mathrm{d}A$ as

$$D_q f(A) = \frac{f(A+\Delta_q A) - f(A)}{\Delta_q A} \tag{6.14}$$

with $\Delta_q A = (q-1)A$ and

$$\frac{\mathrm{d}f(A)}{\mathrm{d}A} = \frac{f(A+\tilde{\delta}_A) - f(A)}{\tilde{\delta}_A} \tag{6.15}$$

with $\tilde{\delta}_A = -\delta_A$, respectively. The structures of the above two expressions are quite similar, though the first one is an operator and the second one is a hyperoperator. Thus, if the condition

$$(\Delta_q A)B = \tilde{\delta}_A B \tag{6.16}$$

is satisfied for some operator B, then the above two expressions (6.14) and (6.15) play the same role when they operate on B from the left-hand side. Condition (6.16) is equivalent to the relation $BA = qAB$. This explains the above mechanism why our quantum analysis is formally related to the q-derivative of a c-number function.

There are many other applications of the general formula (6.5), which will be published in the near future.

7. Deriving systematically quantum corrections

The above integral representations of quantum derivatives and q-derivatives are convenient for studying quantum corrections. In fact, we have

$$\frac{\mathrm{d}^n f(A)}{\mathrm{d}A^n} = f^{(n)}(A) + \sum_{m=1}^{\infty} \frac{n!(-1)^m}{m!} f^{(n+m)}(A) \int_0^1 \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \left(\sum_{j=1}^{n} t_j \delta_j\right)^m \tag{7.1}$$

and

$$D_q^n f(x) = f^{(n)}(x)\frac{[n]_q!}{n!} + \sum_{m=1}^{\infty} \frac{[n]_q!(q-1)^m}{m!} f^{(n+m)}(x) \int_0^1 \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \left(\sum_{j=1}^{n} t_j q^{j-1} x\right)^m$$
$$= f^{(n)}(x) + (q-1)\left\{\frac{n(n-1)}{4} f^{(n)}(x) + \frac{nx}{2} f^{(n+1)}(x)\right\} + \cdots. \tag{7.2}$$

Thus, our quantum analysis based on the hyperoperators A $(=L_A)$ and δ_A is very convenient to study the crossover between classical and quantum behaviours.

8. Conclusion

A refined formulation of quantum analysis has been made to express quantum derivatives in terms of the inner derivation δ_A and this result has been extended to a general case in which the Leibniz rule is satisfied. These results have been shown to be useful in evaluating commutators such as $[f(A), g(B)]$. We have also shown that we can easily derive quantum corrections systematically using our integral representation (2.16) of quantum derivatives. This is a big contrast to the Feynman operator calculus based on Feynman's indices by which quantum derivatives are expressed in such forms as given in the second line of equation (2.8). Higher-order exponential formulae have also been reviewed from the above viewpoint. Multiple integral representations of q-derivatives have been derived, which give a new scheme to study quantum calculus [34]. A simple explanation of the connection between the quantum derivative $\mathrm{d}f(A)/\mathrm{d}A$ and the q-derivative D_q has also been given.

Note added in proof. For some implications of the present quantum analysis, see also the following references.

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